JOURNAL OF APPROXIMATION THEORY 72, 268–277 (1993)

On Local and Controlled Approximation Order

E. J. HALTON

Department of Mathematics and Statistics, University of Central Lancashire, Preston PR1 2TQ, United Kingdom

AND

W. A. LIGHT

Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, United Kingdom

Communicated by Carl de Boor

Received November 27, 1990; accepted in revised form October 24, 1991

This paper deals with approximations to smooth functions by objects which are essentially discrete convolutions. The purpose is to get error estimates for the approximation power of dilates of these convolutions in terms of the dilation parameter h. The paper does this via the well-known Strang–Fix conditions, but there are some important modifications to previous treatments of this theory. In particular, the function used to generate the discrete convolution may or may not have compact support. The paper obtains a coherent theory for both these cases.

1. INTRODUCTION

This paper concerns approximation of smooth functions in $W^k_\infty(\mathbb{R}^n)$ using approximations of the form $\sum_{\psi \in \Psi} \psi * c_\psi$. Here Ψ is a finite collection of functions on $\mathbb{R}^n$ with rapid decay at ∞, each c_ψ is in $l_\infty(\mathbb{Z}^n)$ (i.e., is a bounded function from $\mathbb{Z}^n$ to $\mathbb{R}$), and the convolution is defined by

$$(\psi * c_\psi)(x) = \sum_{v \in \mathbb{Z}^n} \psi(x-v)\, c_\psi(v) \qquad (x \in \mathbb{R}^n).$$

In order to understand the problem, we begin with the space $W^k_\infty(\mathbb{R}^n)$ which consists of all functions $f\colon \mathbb{R}^n \to \mathbb{R}$ for which all derivatives $D^\alpha f$, for $|\alpha| \leqslant k$, exist as members of $L_\infty(\mathbb{R}^n)$. We set

$$\|f\|_{k,\infty} = \sum_{j \leqslant k} |f|_{j,\infty},$$

0021-9045/93 $5.00

where

$$|f|_{j,\infty} = \sum_{|\alpha|=j} \|D^\alpha f\|_\infty.$$

Here $\|\cdot\|_\infty$ is the usual essential supremum norm on $L_\infty(\mathbb{R}^n)$. Define a scaling operator S_h by

$$(S_h f)(x) = f(hx) \qquad (h>0,\ x \in \mathbb{R}^n).$$

It will also be convenient throughout to denote the support of a function in $C(\mathbb{R}^n)$ by supp(f). This is defined as the closure of the set $\{x \in \mathbb{R}^n : f(x) \neq 0\}$.

DEFINITION 1.1. Let Ψ be a finite set of compactly supported functions in $C(\mathbb{R}^n)$. Then Ψ provides local L_∞-approximation of order k, if for each $f \in W^k_\infty(\mathbb{R}^n)$ there exist functions $c^h_\psi \in l_\infty(\mathbb{Z}^n)$ such that, if

$$f_h = S_{1/h} \sum_{\psi \in \Psi} \psi * c^h_\psi,$$

then

(i) $\|f - f_h\|_\infty \leqslant Ah^k |f|_{k,\infty}$, $0 < h < 1$, where the constant A is independent of h and f

(ii) there exists a constant r independent of f and h such that $c^h_\psi(v) = 0$ whenever dist$(vh$, supp$(f)) > r$.

The term "local approximation of order k" is due to de Boor and Jia [1]. The purpose of local approximation was to correct erroneous work by Strang and Fix [6]. To understand the results of de Boor and Jia, it will be convenient to adopt the following form of the Fourier transform:

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(y)\, e^{-ixy}\, dy, \quad \text{where} \quad xy = \sum_{j=1}^n x_j y_j \quad (x, y \in \mathbb{R}^n, f \in L_1(\mathbb{R}^n)).$$

We will also make heavy use of the normalised monomials, V_α, where $V_\alpha(x) = x^\alpha/\alpha!$, $\alpha = (\alpha_1, ..., \alpha_n)$, and $\alpha_i \geqslant 0$, $1 \leqslant i \leqslant n$.

THEOREM 1.2 (de Boor–Jia). *Let Ψ be a finite set of compactly supported functions in $C(\mathbb{R}^n)$. Then Ψ provides local L_∞-approximation of order k if and only if there exists a sequence $\{\psi_\alpha\}_{|\alpha|<k}$ in* span Ψ *such that $\hat{\psi}_0(0) = 1$ and*

$$\sum_{0 \leqslant \beta \leqslant \alpha} V_\beta(-iD)\, \hat{\psi}_{\alpha-\beta}(2\pi v) = 0 \qquad (v \in \mathbb{Z}^n \setminus \{0\},\ |\alpha| < k).$$

Recently, there has been some interest in generalising results like 1.2 to the case where Ψ contains functions of rapid decay at ∞, rather than compact support. A theorem similar to 1.2 was obtained by Cheney and Light [2]. Let E be the subspace of $C(\mathbb{R}^n)$ consisting of all continuous functions $f: \mathbb{R}^n \to \mathbb{R}$ such that

$$\sup_{x \in \mathbb{R}^n} \{ |f(x)| (1 + \|x\|)^{n+k+\lambda} \} < \infty \qquad (k \in \mathbb{Z}_+, \lambda > 0).$$

Then the result of [2] is as follows.

THEOREM 1.3 (Cheney and Light). *Let Ψ be a finite set of functions in E. Then the following are equivalent.*

(i) *There exist functions ψ_α in Ψ ($\alpha \in \mathbb{Z}^n, \alpha \geq 0, |\alpha| < k$) such that $\hat{\psi}_0(0) = 1$ and*

$$\sum_{0 \leq \beta \leq \alpha} V_\beta(-iD)\, \hat{\psi}_{\alpha-\beta}(2\pi v) = 0 \qquad (v \in \mathbb{Z}^n \backslash \{0\}, |\alpha| < k).$$

(ii) *There exist ψ_α in Ψ and b_α in E ($\alpha \in \mathbb{Z}^n$, $\alpha \geq 0$, $|\alpha| < k$) such that the function $\phi = \sum_{|\alpha| < k} \psi_\alpha * b_\alpha$ belongs to E and satisfies*

$$\|f - S_{1/h}(\phi * S_h f)\|_\infty \leq C h^k\, |f|_{k,\infty} \qquad (f \in W^k_\infty(\mathbb{R}^n)).$$

In this theorem the implication (i) $\Rightarrow$ (ii) might seem at first sight to be stronger than that of 1.2, since the approximation order is attained with a particular type of function,

$$u_h = S_{1/h}(\phi * S_h f).$$

Such a function is often called a quasi-interpolant. However, the technique of proof used by de Boor and Jia in fact establishes that the correct approximation order is indeed achieved by a quasi-interpolant. In fact, 1.3 turns out to be weaker than 1.2, since the Fourier transform conditions in (i) are deduced from knowledge that the approximation order is attained by a quasi-interpolant. This in turn allows special information to be gleaned from the structure of the quasi-interpolant, and removes the necessity for a condition akin to localness.

The missing ingredient from 1.3 is a generalization of the idea of local approximation order to functions having rapid decay. In this paper we give an appropriate extension of this concept, and prove the generalisation of 1.2. After completing our work, we became aware of the work of Jia and Lei [4], which is very much in the same spirit as our own. In the next section we describe their contribution as it affects our discussion, and provide the details of our own approach.

2. Local Controlled Approximation Order

In addition to the notation introduced in Section 1, we denote by π_k the subspace of $C(\mathbb{R}^n)$ consisting of polynomials of total degree at most k. The standard multi-index notation is employed throughout our discussions. Most importantly, if $\alpha=(\alpha_1, ..., \alpha_n)\in \mathbb{Z}^n_+$, then $|\alpha|=\sum_{j=1}^n \alpha_j$, and if $x=(x_1, ..., x_n)\in \mathbb{R}^n$ then $x^\alpha=\prod_{j=1}^n x_j^{\alpha_j}$. In addition, for such α, $\alpha!=\prod_{j=1}^n \alpha_j!$, with the usual provision that $0!=1$. The norm on $\mathbb{R}^n$ will always be the supremum norm so that

$$\|x\| = \max_{1\leqslant j\leqslant n} |x_j| \qquad (x\in \mathbb{R}^n).$$

The following linear operators are useful:

$$(Bf)(x)=f(-x), \qquad (T_x f)(y)=f(y-x).$$

It is also sometimes convenient to write $e_y(x)$ for e^{ixy}. The symbol Ψ always denotes a finite set of functions in E, so that each element of Ψ is a continuous function which decays at infinity like $\|x\|^{-n-k-\lambda}$. The chief distinction between our work and that contained in [4] is that we want to prove a theorem for functions in Ψ of rapid decay which, when read for functions of compact support, produces exactly Theorem 1.2 as proved by de Boor and Jia. To do this, we have to be more precise about the behavior at infinity of the functions in the set Ψ. Throughout the paper, n and k denote fixed natural numbers. The notation $E_{r,\lambda}$ denotes the set of functions f in $C(\mathbb{R}^n)$ with the property that

$$|f(x)|\leqslant c(f)(1+\|x\|)^{-n-k-\lambda} \qquad \text{for all} \quad x\in \mathbb{R}^n \text{ with } \|x\|>r.$$

In order that the above notation has content, it is important to allow λ to take the value $+\infty$. We define $E_{r,\infty}$ to be the set of all $f\in C(\mathbb{R}^n)$ such that the support of f is contained in $\{x\colon \|x\|\leqslant r\}$. Note that if $f\in E_{r,\infty}$, then $|f(x)|\leqslant (1+\|x\|)^{-n-k-\lambda}$ for all $\|x\|>r$ and $\lambda>0$. We say that a function f is contained uniformly in E_{r,λ_0} if f lies in $E_{r,\lambda}$ for all $\lambda<\lambda_0$ with the constant $c(f)$ in the above inequality independent of λ.

Definition 2.1. *Let Ψ be a finite set of functions in $E_{r,\lambda}$. Then Ψ provides local, controlled approximation of order k if there exists a constant $A>0$ such that for each $f\in W^k_\infty(\mathbb{R}^n)$ and $0<h<1$ there exist functions $c^h_\psi\colon \mathbb{Z}^n\to \mathbb{R}$, $\psi\in\Psi$, so that*

(i) $\|f-S_{1/h}\sum_{\psi\in\Psi}\psi * c^h_\psi\|_\infty \leqslant Ah^k\,|f|_{k,\infty}$.

(ii) $\|c^h_\psi\|_\infty \leqslant Ah^{-\lambda'}\|f\|_\infty$, *where* $\lambda' = \sup\{\lambda\colon \Psi$ *is contained uniformly in* $E_{r,\lambda}\}$.

(iii) *there exists a constant R independent of h such that if* $\operatorname{dist}(vh, \operatorname{supp} f)>R$, *then* $c^h_\psi(v)=0$ *for all* $\psi\in\Psi$.

The terminology "local controlled approximation" arises by referring to (ii) as a "control" on the size of the coefficients, and to (iii) as a "localness" condition. Note that in the case $\lambda' = \infty$, condition (ii) becomes vacuous.

It will often be convenient to abuse the above terminology slightly. We will, in the case that (i), (ii), and (iii) hold, refer to $S_{1/h} \sum_{\psi \in \Psi} \psi * c_\psi^h$ as providing local, controlled approximation of order k to f. Observe that condition (ii) implies that c_ψ^h belongs to $l_\infty(\mathbb{Z}^n)$ for all $\psi \in \Psi$ and $0 < h < 1$, so that $\psi * c_\psi^h$ is always an absolutely convergent series.

THEOREM 2.2. *Let Ψ be a finite set of functions in $C(\mathbb{R}^n)$ in $E_{r,\lambda}$. Then Ψ provides local controlled approximation of order k if there exists a sequence $\{\psi_\alpha\}_{|\alpha|<k}$ in* span Ψ *such that $\hat{\psi}_0(0) = 1$ and*

$$\sum_{0 \leq \beta \leq \alpha} V_\beta(-iD)\,\hat{\psi}_{\alpha-\beta}(2\pi v) = 0 \qquad (v \in \mathbb{Z}^n \backslash \{0\},\ |\alpha| < k).$$

Proof. Assume the hypotheses of the theorem. Then a combination of [2, 4] shows that there exists a constant A and compactly supported functions $b_\alpha\colon \mathbb{Z}^n \to \mathbb{R}$, $|\alpha| < k$, such that the function $\phi = \sum_{|\alpha|<k} \psi_\alpha * b_\alpha$ belongs to E and satisfies

$$\|f - S_{1/h}(\phi * S_h f)\|_\infty \leq A h^k\, |f|_{k,\infty},$$

for all $f \in W_\infty^k(\mathbb{R}^n)$. Setting $\psi_\alpha = \sum_{\psi \in \Psi} a_\psi^\alpha \psi$, $|\alpha| < k$, gives

$$\begin{aligned}
\phi * S_h f &= \sum_{|\alpha|<k} \psi_\alpha * b_\alpha * S_h f \\
&= \sum_{|\alpha|<k} \left(\sum_{\psi \in \Psi} a_\psi^\alpha \psi \right) * b_\alpha * S_h f \\
&= \sum_{\psi \in \Psi} \psi * \left(\sum_{|\alpha|<k} a_\psi^\alpha b_\alpha \right) * S_h f \\
&= \sum_{\psi \in \Psi} \psi * c_\psi^h,
\end{aligned}$$

where $c_\psi^h = \sum_{|\alpha|<k} a_\psi^\alpha b_\alpha * S_h f$, $h > 0$, $\psi \in \Psi$. Now

$$\begin{aligned}
|c_\psi^h(v)| &\leq \sum_{|\alpha|<k} |a_\psi^\alpha| \sum_{\mu \in \mathbb{Z}^n} |b_\alpha(v-\mu)\, f(h\mu)| \\
&\leq \|f\|_\infty \sum_{|\alpha|<k} |a_\psi^\alpha| \sum_{\mu \in \mathbb{Z}^n} |b_\alpha(v-\mu)| \\
&= \|f\|_\infty \sum_{|\alpha|<k} |a_\psi^\alpha| \sum_{\mu \in \mathbb{Z}^n} |b_\alpha(\mu)|.
\end{aligned}$$

By redefining $A := \max\{A, \max_{\psi \in \Psi} \sum_{|\alpha| < k} |a_\psi^\alpha| \sum_{\mu \in \mathbb{Z}^n} |b_\alpha(\mu)|\}$ it follows immediately that $|c_\psi^h(\nu)| \leqslant A \|f\|_\infty$ for $\nu \in \mathbb{Z}^n$. Since for $0 \leqslant h \leqslant 1$, $h^{-\lambda'} \geqslant 1$, it is trivial that $\|c_\psi^h\|_\infty \leqslant A h^{-\lambda'} \|f\|_\infty$ for $\psi \in \Psi$.

Now suppose each b_α, $|\alpha| < k$, is supported in a ball of radius ρ. Then

$$|c_\psi^h(\nu)| \leqslant \sum_{|\alpha| < k} |a_\psi^\alpha| \sum_{\mu \in \mathbb{Z}^n} |b_\alpha(\mu) f(h(\nu - \mu))|$$

$$= \sum_{|\alpha| < k} |a_\psi^\alpha| \sum_{\|\mu\| \leqslant \rho} |b_\alpha(\mu) f(h(\nu - \mu))|.$$

If $\operatorname{dist}(\nu h, \operatorname{supp} f) > \rho$ then for $0 < h < 1$ it follows that $h\nu - h\mu \notin \operatorname{supp} f$ for $\|\mu\| \leqslant \rho$. Hence $c_\psi^h(\nu) = 0$ whenever $\operatorname{dist}(\nu h, \operatorname{supp} f) > \rho$, and $0 < h < 1$. ∎

We now want to derive the reverse implication to 2.2. Our approach is to follow quite closely the method of proof given in [2]. Since many aspects of this proof are already well-documented in [2], we only indicate the new features needed in the present case. The following result seems critical in all treatments of this argument.

LEMMA 2.3. *Let Ψ be a finite subset of $E_{r,\lambda}$. Let $f \in W_\infty^k(\mathbb{R}^n)$ be a function having compact support. Let $f_h = S_{1/h}(\sum_{\psi \in \Psi} \psi * c_\psi^h)$ provide local, controlled approximation of order k to f. Then there exist constants C and ρ independent of h such that*

$$|f_h(x)| \leqslant C h^k \|x\|^{-n-k-\lambda} \qquad \textit{for} \quad \|x\| > 2\rho.$$

Proof. Since f_h produces local, controlled approximation of order k to f, and f has compact support, we can assume that $f(x) = 0$ for all $\|x\| > \rho$, and $c_\psi^h(\nu) = 0$ for all $\|\nu h\| > \rho$. Then, for any $x \in \mathbb{R}^n$,

$$|f_h(x)| = \left| S_{1/h} \sum_{\psi \in \Psi} (\psi * c_\psi^h)(x) \right|$$

$$\leqslant \sum_{\psi \in \Psi} \sum_{\nu \in \mathbb{Z}^n} \left| \psi\left(\frac{x}{h} - \nu\right) c_\psi^h(\nu) \right|$$

$$\leqslant \sum_{\psi \in \Psi} \sum_{\|\nu h\| \leqslant \rho} \left| \psi\left(\frac{x}{h} - \nu\right) \right| |c_\psi^h(\nu)|.$$

Suppose now $\lambda = \infty$. By increasing ρ if necessary (so that $\rho > r$), we can assume that $\operatorname{supp} \psi \subset \{x \in \mathbb{R}^n : \|x\| \leqslant \rho\}$ for $\psi \in \Psi$. If $\|x\| > 2\rho$, $\|\nu h\| \leqslant \rho$, and $0 < h < 1$, then

$$\left\| \frac{x}{h} - \nu \right\| \geqslant \left\| \frac{x}{h} \right\| - \|\nu\| \geqslant \frac{2\rho}{h} - \frac{\rho}{h} = \frac{\rho}{h} \geqslant \rho.$$

Hence we see that $f_h(x)=0$ for $\|x\|>2\rho$, and we may take $C=0$. If $\lambda<\infty$ then, using property (ii) of 2.1,

$$|f_h(x)| \leqslant Bh^{-\lambda} \sum_{\psi\in\Psi} \sum_{\|\nu h\|<\rho} \left(1+\left\|\frac{x}{h}-\nu\right\|\right)^{-n-k-\lambda}$$

$$= Bh^{n+k} \sum_{\psi\in\Psi} \sum_{\|\nu h\|<\rho} (h+\|x-\nu h\|)^{-n-k-\lambda}.$$

In the above sum we have, for $\|x\|>2\rho$,

$$h+\|x-\nu h\| \geqslant \|x\|-\|\nu h\| \geqslant \|x\|-\rho \geqslant \|x\|/2,$$

and so

$$|f_h(x)| \leqslant Bh^{n+k} \sum_{\psi\in\Psi} \sum_{\|\nu h\|\leqslant\rho} (\|x\|/2)^{-n-k-\lambda}$$

$$\leqslant B_1 h^{n+k} \|x\|^{-n-k-\lambda} \sum_{\psi\in\Psi} \sum_{\|\nu h\|\leqslant\rho} 1$$

$$\leqslant B_2 h^{n+k}(\rho/h)^n \leqslant B_3 h^k \|x\|^{-n-k-\lambda}. \quad ∎$$

LEMMA 2.4. *Assume the hypotheses of* 2.3. *Then*

(i) $\|f-f_h\|_\infty$ *is* $\mathcal{O}(h^k)$ *as* $h\to 0$ *implies* $\|V_\alpha(D)(f-f_h)\|_\infty$ *is* $\mathcal{O}(h^k)$ *for* $|\alpha|<k$

(ii) $\hat{f}_h = h^n \sum_{\psi\in\Psi} S_h\hat{\psi} \sum_{\nu\in\mathbb{Z}^n} e_{-\nu h} c_\psi^h(\nu)$, *for each* $h>0$.

The proofs of these statements may be found in [2]. We now introduce the function

$$u(x)=\prod_{i=1}^n M_{k+1}(x_i) \qquad (x\in\mathbb{R}^n),$$

where M_{k+1} is a univariate B-spline of order k, having the properties that

$$\hat{u}(x)=\prod_{i=1}^n \left[\frac{\sin(x_i/2)}{(x_i/2)}\right]^{k+1}$$

and

$$[V_\alpha(D)\hat{u}](x/h)=o(h^k) \qquad (x\neq 0,\ h\to 0,\ |\alpha|<k).$$

It follows from [2] that if $u_h=S_{1/h}(\sum_{\psi\in\Psi}\psi * c_\psi^h)$ provides local, controlled approximation of order k to u, then

$$\lim_{h\to 0} h^n \sum_{\psi\in\Psi} \sum_{0\leqslant\beta\leqslant\alpha} [V_\beta(iD)\hat{\psi}](2\pi\mu) \sum_{\nu\in\mathbb{Z}^n} V_{\alpha-\beta}(\nu)\, c_\psi^h(\nu)=0$$
$$(|\alpha|<k,\ \mu\in\mathbb{Z}^n\setminus\{0\}). \qquad (1)$$

Our second theorem now follows. Its proof is modelled closely on the one in [1].

THEOREM 2.5. *Let Ψ be a finite subset of $E_{r,\lambda}$ which provides local, controlled approximation of order k. Then there exists a sequence $\{\psi_\alpha\}_{|\alpha|<k}$ in Ψ such that $\hat{\psi}_0(0)=1$ and*

$$\sum_{0\leqslant\beta\leqslant\alpha} V_\beta(-iD)\,\hat{\psi}_{\alpha-\beta}(2\pi v)=0, \qquad (v\in\mathbb{Z}^n\backslash\{0\},\ |\alpha|<k).$$

Proof. We work entirely with the function u defined previously. Firstly, there must exist a $\psi\in\Psi$ such that $\hat{\psi}(0)\neq 0$, because if not, then 2.4(ii) gives

$$\hat{u}_h(0)=h^n\sum_{\psi\in\Psi}(S_h\hat{\psi})(0)\sum_{v\in\mathbb{Z}^n} e_{-vh}(0)\,c^h_\psi(v)=h^n\sum_{\psi\in\Psi}\hat{\psi}(0)\sum_{v\in\mathbb{Z}^n}c^h_\psi(v)=0.$$

Then 2.4(i) show that $\|\hat{u}-\hat{u}_h\|_\infty=\mathcal{O}(h^k)$ and since $\hat{u}(0)=1$, this contradicts $\hat{u}_h(0)=0$ as $h\to 0$.

Since u_h is unchanged if each element in Ψ is replaced by elements in span Ψ, as long as the overall span is identical, we may assume that there is a $\chi\in\Psi$ with $\hat{\chi}(0)=1$ and that $\hat{\psi}(0)=0$ for all ψ in Ψ with $\psi\neq\chi$. Then, from 2.4(ii),

$$1=\lim_{h\to 0}\hat{u}_h(0)=\lim_{h\to 0}h^n\sum_{\psi\in\Psi}\hat{\psi}(0)\sum_{v\in\mathbb{Z}^n}c^h_\psi(v)=\lim_{h\to 0}h^n\sum_{v\in\mathbb{Z}^n}c^h_\chi(v).$$

Let S be the set of all vectors $w=(w_{\psi,\gamma})$ where $\psi\in\Psi$ and $|\gamma|<k$, satisfying

$$\lim_{h\to 0}\sum_{\psi\in\Psi}\sum_{|\gamma|<k}w_{\psi,\gamma}h^n\sum_{v\in\mathbb{Z}^n}c^h_\psi(v)\,V_\gamma(v)=0.$$

We claim $S^\perp$ contains the vector w' with $w'_{\chi,0}=1$. If not, then $v_{\chi,0}=0$ for all $v\in S^\perp$. It then follows that $(S^\perp)^\perp$ contains the vector $\delta_{\chi,\psi}\delta_{0,\sigma}$, $\psi\in\Psi$, $|\sigma|<k$, which in turn lies in S. This gives rise to the contradiction

$$\begin{aligned}0&=\lim_{h\to 0}\sum_{\psi\in\Psi}\sum_{|\sigma|<k}\delta_{\chi,\psi}\delta_{0,\sigma}h^n\sum_{v\in\mathbb{Z}^n}c^h_\psi(v)\,V_\sigma(v)\\&=\lim_{h\to 0}h^n\sum_{v\in\mathbb{Z}^n}c^h_\chi(v)\\&=1.\end{aligned}$$

Define

$$\psi_\gamma=(-1)^{|\gamma|}\sum_{\psi\in\Psi}w'_{\psi,\gamma}\psi, \qquad |\gamma|<k.$$

Then

$$\hat{\psi}_0(0) = \sum_{\psi \in \Psi} w'_{\psi,0}\hat{\psi}(0) = w'_{\chi,0}\hat{\chi}(0) = 1.$$

Furthermore, by Eq. (1),

$$\lim_{h \to 0} h^n \sum_{\psi \in \Psi} \sum_{0 \leqslant \beta \leqslant \alpha} [V_\beta(iD)\hat{\psi}](2\pi\mu) \sum_{v \in \mathbb{Z}^n} V_{\alpha-\beta}(v)\, c^h_\psi(v) = 0,$$

for $|\alpha| < k$ and $\mu \neq 0$. Making the change of variable $\alpha - \beta = \gamma$ gives

$$\lim_{h \to 0} h^n \sum_{\psi \in \Psi} \sum_{0 \leqslant \gamma \leqslant \alpha} [V_{\alpha-\gamma}(iD)\hat{\psi}](2\pi\mu) \sum_{v \in \mathbb{Z}^n} V_\gamma(v)\, c^h_\psi(v) = 0,$$

for $|\alpha| < k$ and $\mu \neq 0$. This shows that

$$\{[V_{\alpha-\gamma}(iD)\hat{\psi}](2\pi\mu)\}_{\psi,\gamma}$$

is in S for $|\alpha| < k$ and $\mu \neq 0$. Hence,

$$\sum_{\psi \in \Psi} \sum_{|\gamma| < k} w'_{\psi,\gamma}[V_{\alpha-\gamma}(iD)\hat{\psi}](2\pi\mu) = 0, \qquad \mu \neq 0,\ |\alpha| < k.$$

Finally, for $\mu \neq 0$,

$$\begin{aligned}
&\sum_{0 \leqslant \beta \leqslant \alpha} V_\beta(-iD)\,\hat{\psi}_{\alpha-\beta}(2\pi\mu) \\
&\quad = \sum_{0 \leqslant \beta \leqslant \alpha} (-1)^{|\beta|}\, V_\beta(iD)\left[(-1)^{|\alpha-\beta|} \sum_{\psi \in \Psi} w'_{\psi,\alpha-\beta}\hat{\psi}\right](2\pi\mu) \\
&\quad = (-1)^{|\alpha|} \sum_{\psi \in \Psi} \sum_{0 \leqslant \beta \leqslant \alpha} w'_{\psi,\alpha-\beta}[V_\beta(iD)\hat{\psi}](2\pi\mu) \\
&\quad = (-1)^{|\alpha|} \sum_{\psi \in \Psi} \sum_{|\gamma| < k} w'_{\psi,\gamma}[V_{\alpha-\gamma}(iD)\hat{\psi}](2\pi\mu) \\
&\quad = 0,
\end{aligned}$$

for $\mu \neq 0$ and $|\alpha| < k$. ∎

References

1. C. de Boor and R. Q. Jia, Controlled approximation and a characterisation of the local approximation order, *Proc. Amer. Math. Soc.* **95** (1985), 547–553.
2. E. W. Cheney and W. A. Light, Quasi-interpolation with base functions having non-compact support, *Constr. Approx.* **8** (1992), 35–48.

3. W. Dahmen and C. A. Micchelli, On the approximation order from certain multivariate spline spaces, *J. Austral. Math. Soc. Ser. B* **26** (1984), 233–246.
4. R.-Q. Jia and J. Lei, Approximation by multiinteger translates of functions having non-compact support, *J. Approx. Theory*, to appear.
5. W. A. Light, Recent developments in the Strang–Fix theory for approximation orders, *in* "Curves and Surfaces" (P. J. Laurent, A. Le Méhauté, and L. L. Schumaker, Eds.), pp. 285–292, Academic Press, New York, 1990.
6. G. Strang and G. Fix, A Fourier analysis of the finite-element variational method, *in* "Constructive Aspects of Functional Analyses" (G. Geymonat, Ed.), pp. 793–840, C.I.M.E., 1973.